I. *A Discourse concerning the Methods of Approximation in the Extraction of Surd Roots. By* John Wallis, *S. T. D. and* Savilian *Professor of Geometry at* Oxford.

THE several Methods of Approximation, which have been mentioned of late Years, for Extracting the Roots of Simple or Affected Equations, gives me occasion of saying somewhat of that Subject.

It is agreed by all, (and, I think, demonstrated by the *Greeks* long ago) that if a Number proposed be not a true Square, it is in vain for to hope for a just Quadratick Root thereof, explicable by Rational Numbers, Integers, or Fracted. And therefore, in such cases, we must content our selves with Approximations (somewhat near the truth) without pretending to Accuracy.

And so, for the Cubick Root, of what is not a perfect Cube. And the like for Superiour Powers.

Now the Ancients (being aware of this) had their Methods of Approximation in such cases; whereof some have been derived down even to this day. Of which we shall speak more anon.

But since the Methods of Decimal Fractions (as they are now wont to be called) have come into Practice, it hath been usual to prosecute such Extractions (beyond the place of Unites) in the places of Decimal Parts to what Accuracy we please; whereby the former Methods of Approaches have been (not so much forgotten, as) neglected.

Not that if such Approaches by Decimals were always the most speedy, or the most exact; (for no Man doubts but that $\frac{1}{8}$ is a more Simple, and more Intelligible Notation of that Quantity, than 0.125, or $\frac{125}{1000}$: And $\frac{1}{3}$, not only a more Brief, but a more Accurate designation of the Square Root of $\frac{1}{9}$, than 0.333333, *&c.*) But, because Fractions reduced to the Decimal form, are more convenient for subsequent Operations, when there is occasion for a further Progress.

Mr. *Newton*'s Method of Approximation for the Extracting Roots, even of Affected Equations, I have given some Account

count of in my *English Algebra*; and somewhat more fully in the *Latin* Edition: where I give an Account also of Mr. *Raphson's* Method; which I need not here repeat, because it is to be seen there.

Since which time, Monsieur *De L'agny* hath published his Method of Approximation, principally for single Equations, or Extracting the Root of a single Power.

And Mr. *Halley* hath since improved this Method, with a further Advantage; especially as to Affected Equations.

But I need not repeat either of them, because they are both publisht. That of *De L'agny* in a Treatise by it self; and that of Mr. *Halley*, in the *Philos. Transact.* Numb. 210, for the Month of *May*, 1694.

These may all, or any of them, be of good use (each in their own way) for making more speedy Approaches, and by greater Leaps, in many cases, than *Vieta's* Method (prosecuting the Extraction in Decimal Parts) pursued and improved by Mr. *Oughtred* and Mr. *Harriot* of our own, and by others abroad; especially as to Simple Equations, if we suppose such Extractions to be pursued to the full extent.

As for instance, if we would Extract the Root of an imperfect Sursolid, (or a Power of five Dimensions) to have its Root true as far as the sixth place of Decimal Parts. In order to this, we are to add (or suppose to be added) six Punctations of Cyphers, (or, six times five Cyphers, that is Thirty Cyphers) beyond the place of Unites in the Number proposed. If now we pursue the whole Operation to the utmost of those Thirty places, the Work would be long and tedious.

But if we make use of Mr. *Oughtred's* Expedient, (for Multiplication, Division, Extraction of Roots, and other like Operations,) by neglecting so much of this long Process, as is afterward to be cut off and thrown away as useless, (which, I think, is generally practised) the Work will be much abbridged, and the Advantage of the other Methods much less considerable.

That is, If (for ascertaining the sixth place of Decimal Parts) we add six Cyphers (instead of thirty), and one or two more (the better to secure what from the consequent places may in the Operation be transmitted hither;) and pursue the Operation thus far, neglecting the following places

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(which are not likely to influence the Figure due to the ſixth place of Decimal Parts of the Root ſought:) this long Proceſs will be much ſhortned.

And if we further conſider, what Preparative Operations are to be made in ſome of thoſe other Methods, before we come to the preſcribed Diviſion for giving the Root deſired; the Advantage (though conſiderable) will not be ſo great as may at firſt be apprehended; eſpecially as to Affected Equations, in which the Parodical Powers have great Coefficients. As will ſoon appear in Practice, if we come to apply them to particular caſes.

But, without diſparaging theſe Methods (which are really conſiderable, and well worthy encouragement) that which I here intend, is, to ſhew the true Foundation of the Methods uſed by the Ancients, (however ſince neglected) and the juſt Improvement of them. Which though Anciently ſcarce applyed beyond the Quadratick, or perhaps the Cubick Root, (for with the Higher Powers they did not much trouble themſelves) yet are equally applicable (by due adjuſtments) to the Superiour Powers alſo.

I ſhall begin with the Square Root: For which the Ancient Method is to this purpoſe.

From the propoſed Non-quadrate (ſuppoſe N) ſubtract (in the uſual manner) the greateſt Square in Integers therein contained (ſuppoſe Aq.) The remainder (ſuppoſe $B = 2AE + Eq$) is to be the Numerator of a Fraction, for deſigning the near value of E (the remaining part of the Root ſought $A + E = \surd N$,) whoſe Denominator or Diviſor is to be 2 A (the double Root of the ſubtracted Square) or $2A + 1$ (that double Root increaſed by 1) the true value falling between theſe two; ſometime the one, ſometime the other, being neareſt to the true value. But (for avoiding of Negative Numbers) the latter is commonly directed.

This Method Monſieur *De L'agny* affirms to be more than 200 Years old: And it is ſo; for I find it in *Lucas Pacciolus* (otherwiſe called *Lucas de Burgo*, or *de Burgo Sancti Sepulchri*) Printed at *Venice* in the Year 1494 (if not even ſooner than ſo, for I find there have been ſeveral Editions of it.) And how much older than ſo, I cannot tell: For he doth not deliver it as a new Invention of his own, but as a received Practice, and derived from the *Moors* or *Arabs*, from whom they had their *Algorism*, or Practice of Arithmetick by the Ten Numeral Figures now in use.

And it is continued down hitherto in Books of Practical Arithmetick in all Languages, which teach the Extraction of the Square Root, and (therein) this Method of Approximation, in case of a Non-quadrate.

The true ground of the Rule is this: Aq being (by Construction) the greatest Integer Square contained in N, 'tis evident that E must be less than 1; (otherwise not Aq, but the Square of $A+1$, or some greater than so, would be the greatest Integer Square contained in N.) Now if the Remainder $B=2AE+Eq$ be divided by 2 A, the Result will be too great for E, (the Divisor being too little; for it should be $2A+E$, to make the Quotient E.) But if (to rectifie this) we diminish the Quotient, by increasing the Divisor, adding 1 to it, it then becomes too little; because the Divisor is now too big. For (E being less than 1) $2A+1$ is more than $2A+E$; and therefore too big.

As for instance; If the Non-quadrate proposed be $N=5$, the greatest Integer Square therein contained is $Aq=4$ (the Square of $A=2$:) which being subtracted, leaves $N-Aq=5-4=1=B=2AE+Eq$. Which divided by $2A=4$, gives $\frac{1}{4}$: But divided by $2A+1=4+1=5$, gives $\frac{1}{5}$. That too great, and this too liitle for E. And therefore the true Root $(A+E=\surd N)$ is less than $2\frac{1}{4}=2.25$, but greater than $2\frac{1}{5}=2.2$: And this was Anciently thought an Approach near enough.

If this Approach be not now thought near enough, the same Process may be again repeated; and that as oft as is thought necessary.

Take now for A, $2\frac{1}{5}=2.2$, whose Square is $4.84=Aq$, (now considered as an Integer in the second place of Decimal Parts.) This subtracted from 5.00, (or, which is the same, 0.84 the excess of this Square above the former, from 1 which was then the remainder,) leaves a new remainder $B=0.16$: which divided by $2A=4.4$, gives $\frac{.16}{4.40}=\frac{2}{55}=0.03636+$, too much. But divided by $2A+1=4.5$, it gives $\frac{.16}{4.50}=\frac{8}{225}=0.03555+$, too little. The true Value (between these two) being 2.236 *proxime*, whose Square is 4.999696.

 If

If this be not thought near enough, Subſtract this Square from 5.000000: The Remainder B = 0.000304, divided by 2 A = 4.472, or by 2 A + 1 = 4.473, gives (either way) 0 000068 —; which added to A = 2.236, makes 2.236068 —, ſomewhat too big; but 2.236067 + would be much more too little.

Which gives us the Square Root of 5, adjuſted to the ſixth place of Decimal Parts, at three ſteps. And by the ſame Method, if it be thought needful, we may proceed further.

It were eaſie to compound the Proceſs of two or more ſteps into one, and give (for the Rule) the Reſult of ſuch Compoſition. Which would make it ſeem more Intricate and Myſterious, to amuſe the Reader: But I chooſe to make it as plain as I can (and therefore keep the ſeveral ſteps ſeparate,) that the Reader (for his ſatisfaction) may clearly ſee the true ground of the Proceſs. But of this enough.

Proceed we now to the Cubick Root. For which (conſonant to the Quadratick) the Rule is this:

From the Non-Cubick propoſed, (ſuppoſe N,) ſubtract the greateſt Cube in Integers therein contained, (ſuppoſe Ac:) the remainder (ſuppoſe B = 3 A q E + 3 A E q + E c,) is to be the Numerator of a Fraction for deſigning the value of E, (the remaining part of the Root ſought A + E = √c N.) To this Numerator, if (for the Denominator or Diviſor) we ſubjoyn 3 A q, the Reſult will certainly be too great for E, becauſe the Diviſor is too little: (For it ſhould be 3 A q + 3 A E + E q, to give the true value of E.) If, for the Diviſor, we take 3 A q + 3 A + 1, it will certainly be too little, becauſe the Diviſor is too great. (For E, by conſtruction, is leſs than 1.) It muſt therefore (between theſe limits) be more than this latter. And therefore this latter Reſult being added to A, will give a Root whoſe Cube may be ſubtracted from the Non-Cubick propoſed, in order to another ſtep.

This Approach I find in *Wingate's* Arithmetick, Publiſhed in the Year 1630, and muſt therefore be at leaſt ſo old; how much older I cannot tell.

But if, for the Diviſor, we take 3 A q + 3 A, (or even leſs than ſo) the Reſult may be too great; or (in caſe B be ſmall) it may be too little, (and oft is ſo.)

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Which comes to paſs from hence; becauſe E (by Conſtruction) is leſs than 1; and therefore 3 A E leſs than 3 A; and perhaps ſo much as that the addition of E q will not redreſs it. And when it ſo happens, 3 A q + 3 A, is a better Diviſor than 3 A q + 3 A + 1, (or even ſomewhat leſs than either.) But becauſe it doth not always ſo happen (though for the moſt part it doth) the Rule doth rather direct the other; as which doth certainly give a Root leſs than the true value, whoſe Cube may always be ſubtracted from the Non-Cubick propoſed. The deſign being to have ſuch a Cube as (being ſubtracted) may leave another B, to be ordered in like manner for a new Approach.

But, for the moſt part, 3 A q may be ſafely taken for the Diviſor. For, though the Reſult will then be ſomewhat too big, yet the exceſs may be ſo ſmall, as to be neglected; or, at leaſt, we may thence eaſily judge what Number (ſomewhat leſs than it) may be ſafely taken. And if we chance to take it ſomewhat too big, the Inconvenience will be but this, that B for the next ſtep will be a Negative. Of which caſe we ſhall ſpeak anon.

Thus, for inſtance; if the Non-Cube propoſed be $9 = N$. The greateſt Integer Cube therein contained is $8 = A c$, (whoſe Cubick Root is $A = 2$.) Which Cube ſubtracted, leaves $9 - 8 = 1 = B = 3\,A q E + 3\,A E q + E c$. This divided by $3 Aq = 12$, gives $\frac{1}{12} = 0.08333 +$, too big for E. But the ſame divided by $3 Aq + 3 A + 1 = 12 + 6 + 1 = 19$, gives $\frac{1}{19} = 0.05263 +$, too little. Or if but by $3 Aq + 3 A = 12 + 6 = 18$, it gives $\frac{1}{18} = \frac{5}{90} = 0.05555 +$, yet too little. For the Cube of $A + 0.06, = 2.06$, is but $8.742 -$, which is ſhort of 9. And ſo much ſhort of it, that we may ſafely take 2.07 as not too big: Or perhaps 2.08, (which if it chance to be too big, it will not be much too big; of which caſe we are to ſpeak anon:) And, upon tryal, it will be found not too big; for the Cube of 2.08, is but 8.998912.

If this firſt ſtep be not near enough: This Cube ſubtracted from 9.000000, leaves a new $B = 0.001088$, which divided by $3 Aq = 12.9796$, gives $0.000084 -$; which will be ſomewhat too big, but not much. (For E is now ſo ſmall, as that 3 AE may be ſafely neglected, and Eq much more.) So that if to 2.08, we add $0.000084 -$, the Reſult 2.080084 will be too big, but 2.080083 will be more too little. (As will appear if we take the Cube of each.) So that either of them, at the second step, gives the true Root within an Unite in the sixth place of Decimal Parts.

But when I say, *Taking the Cube of each*, (which I do, that the thing may be more clearly apprehended) it is not necessary that we trouble our selves with the whole Cube. For, Ac being already subtracted, for finding B=3AqE+3AEq+Ec, we have no more to try, but whether 3AqE+3AEq+Ec be greater or less than B, according as we take 0.000084, or 0.000083, for E.

Which may conveniently be done in this manner: Take 3A+E, and Multiply this by E, (or E by it) so have we 3AE+Eq. To this add 3Aq, and Multiply the whole by E, (so have we 3AqE+3AEq+Ec,) to see whether this be greater or less than B.

That is, in the present case, if we take E=0.000084, and add to this 3A=6.24, then is 6.240084=3A+E. This multiplied by E=0.000084, is 3AE+Eq=0.000524+. To which if we add 3Aq=12.9792, it is 3Aq+3AE+Eq =12.979724. Which multiplied again by E=0.000084, is 0.0010902+=3AqE+3AEq+Ec, which is more than B=0.001088.

But if we take E=0.000083, and proceed as before, we shall have 3AqE+3AEq+Ec=0.001077+, which is less than B=0.001088. And therefore (if we subtract that from this) the Remainder, 0.000011, will be another B for the next step, if we please to proceed further.

Hitherto I have pursued the Method most affected by the Ancients, in seeking a Square or Cube (and the like of other Powers) always less than the just value, that it might be subtracted from the Number proposed, leaving B a Positive Remainder; thereby avoiding Negative Numbers.

But since the Arithmetick of Negatives is now so well understood, it may in this (and other Operations of like Nature) be advisable, to take the next greater (in case that be nearer to the true value) rather than the next lesser. Of which I took notice in my *Commercium Epistolicum*, *Epist*. 19. *Jan*. 2. 165⅞. in a case more intricate than this is. And which I elsewhere advise, in seeking the *Greatest Common Divisor* of two Numbers, in order to the abbridging a Fraction, or otherwise.

Accord-

According to this Notion, for the Square Root of 5, I would ſay, it is (2 +) ſomewhat more than 2 ; and enquire, How much more? But for the Square Root of 8, I would ſay, it is (3 —) ſomewhat leſs than 3 ; and inquire, How much leſs? Taking (in both caſes) that which is neareſt to the juſt Value.

Thus, in the Cubick Root before us; I would take for E (in the laſt Enquiry) 0.000084 — (where , for the next ſtep, we have B = — 0.000002,) rather than 0.000083 + (where, for the next ſtep, we ſhould have B = + 0.000011.) In the latter caſe, we are to Divide B = + 0.000011, by 3Aq = 12.980236 —, to find (by the Quotient) how much is to be added to 0.000083. In the other caſe, we are to Divide B = + 0.000002, by 3Aq = 12.980248, to find (by the Quotient) what is to be abated of 0.000084. So have we

$$\frac{0.000011}{12.980236} = 0.00000085 + \text{ to be added to } 6.240083 : \text{ Or}$$

$$\frac{-0.000002}{12.980248} = 0.00000015 + \text{ to be abated of } 6.240084.$$

(Or it may ſuffice, in either, to Divide by 12.98 +, or even by 13. —, without being incumbred with a long Diviſor) either of which gives us, for the Root ſought, 2.08008385 *proxime*. True (at the third ſtep) to the Eighth place of Decimal Parts. And if this be not near enough, the Cube of this, compared with the Number propoſed, will give us another B for the next ſtep. And ſo onwards as far as we pleaſe.

Now, what is ſaid of the Cube, is eaſily applicable to the Higher Powers.

I ſhall omit that of the Biquadrate ; becauſe here perhaps it may be thought moſt adviſable, to Extract the Square Root of the Number propoſed ; and then the Square Root of that Root.

But if we would do it at once, we are from N (the Number propoſed, being not a Biquadrate) to Subtract Aqq (the greateſt Biquadrate contained in it) to find the Remainder B = 4AcE + 6AqEq + 4AEc + Eqq. Which Remainder, if we Divide by 4Ac, the Quotient will certainly be too big for E,(though perhaps not much:) If by 4Ac + 6Aq + 4A + 1, it will certainly be too little (for reaſons before mentioned.) And we are to uſe our diſcretion in taking ſome in-

termediate

termediate Number. And if we chance not to hit on the neareſt, the Inconvenience will be but this, that our Leap will not be ſo great as otherwiſe it might be. Which will be rectified by another B at the next ſtep.

For the Surſolide (of five Dimenſions) we are, from N (the Number propoſed, being not a perfect Surſolide) to Subtract Aqc (the greateſt Surſolide therein contained) to find the Remainder B $=$ 5 AqqE $+$ 10 AcEq $+$ 10 AqEc $+$ 5 AEqq $+$ Eqc. Which (as before) if we Divide by 5Aqq, the Reſult will be ſomewhat too big, (becauſe the Diviſor is too little:) If by 5Aqq$+$10Ac$+$10Aq$+$5A$+$1, the Reſult will certainly be leſs than the true E. The juſt value of E being ſomewhat between theſe two, where we are to uſe our diſcretion, what Intermediate Number to take. Which according as it proves too great or too little, is to be rectified at the next ſtep.

If, to direct us in the choice of ſuch intermediate Number, we ſhould Multiply Rules or Precepts for ſuch choice, the Trouble of obſerving them, would be more than the Advantage to be gained by it. And, for the moſt part, it will be ſafe enough (and leaſt trouble) to Divide by 5Aqq, which gives a Quotient ſomewhat too big: Which we may either Rectifie at Diſcretion (by taking a Number ſomewhat leſs) or proceed to another B, (Affirmative or Negative, as the caſe ſhall require) and ſo onward to what exactneſs we pleaſe. (Which is, for ſubſtance, in a manner coincident with Mr. *Raphſon*'s Method, even for Affected Equations.

Thus, in the preſent caſe; If the Number propoſed be N $=$ 33, then is Aqc$=$32, and B$=$33$-$32$=$1$=$5AqqE $+$ 10 AcEq $-$ 10 AqEc $+$ 5 AEqq $+$ Eqc. Which if we Divide by 5Aqq $= 5 \times 16 = 80$, the Reſult $\frac{1}{80}$$=$0.0125, is ſomewhat too big for E, but not much. And if we examine it, by taking the Surſolide of 2.0125, or of 2$\frac{1}{80}$, we ſhall find a Negative B (for the next ſtep) but not very conſiderable. Or if we think it conſiderable, we may proceed further to another ſtep, or more than ſo.

The like Method may be applied (and with more Advantage) in the Higher Powers, according as the Compoſition of each Power requires.

And the ſame Method may be of uſe (with good Advantage) in long Numbers (if duly applied) even before we come

come to the place of Unites, for the ſame will equally hold there alſo. Which the Reader may eaſily apprehend without a long Diſcourſe upon it.

How far this Method may be coincident with ſome of thoſe before mentioned, I do not trouble my ſelf to enquire; nor whether, or for what cauſes, all or any of thoſe may be more eligible. My deſign being only to ſhew the true Natural ground, from whence ſuch Rules of Approach are (or might have been) derived; and by which (if there be occaſion) they may be examined. And if I have done this, it is what I did propoſe.

In Affected Equations (eſpecially where the Coefficients are great, and ſome Affirmatives, others Negatives,) the Caſes will be more perplexed. And to Multiply Rules for each Caſe, would (I conceive) increaſe the Trouble, with no great Advantage. Which therefore I leave to the Prudence of each (as occaſion ſhall require) to take ſome Intermediate, between a greater and a leſſer. Or if they pleaſe to accommodate that above mentioned (out of *Commerc. Epiſtol.*) to the preſent caſe, which is there applied to a Caſe not leſs intricate. Or to make uſe of ſome of the Methods delivered by others to this purpoſe. Where this (withal) is to be conſidered, That ſuch Affected Equations are capable of more Roots than one, according to the Number of Dimenſions to which they ariſe.